

# ENERGY AND UNIQUENESS IN THE NAVIER-STOKES EQUATIONS

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**ABSTRACT.** In this paper, we improve new conditions for uniqueness results of weak solutions for the 3D Navier-Stokes equations. The proof has a simple geometrical and physics interpretation it expressed in terms of the enstrophy and the Hausdorff measure of the level set of solutions of the 3D Navier Stokes equations. We apply our result to the 2D Navier Stokes equations.

## 1. INTRODUCTION

Two of the profound open problems in the theory of three dimensional viscous flow are the unique solvability theorem for all time and the regularity of solutions. For the three-dimensional Navier-Stokes system weak solutions of problem are known to exist by a basic result by J. Leray from 1934 [6], it is not known if the weak solution is unique or what further assumption could make it unique only the uniqueness of weak solutions remains as an open problem. Using the enstrophy we give new quantities to control the uniqueness of weak solutions of the three-dimensional Navier-Stokes equations. We prove that, both solutions have the same energy are equal. This paper yields an interesting method for solving uniqueness problem, where Gronwall's Lemma is unable to show the uniqueness of solutions.

## 2. PRELIMINARY

We denote by  $H_{per}^m(\Omega)$ , the Sobolev space of  $L$ -periodic functions endowed with the inner product

$$(u, v) = \sum_{|\beta| \leq m} (D^\beta u, D^\beta v)_{L^2(\Omega)} \text{ and the norm } \|u\|_m = \sum_{|\beta| \leq m} (\|D^\beta u\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$$

We define the spaces  $V_m$  as completions of smooth, divergence-free, periodic, zero-average functions with respect to the  $H_{per}^m$  norms.  $V'_m$  denote the dual space of  $V_m$  and  $V$  denote the space  $V_0$ .

We denote by  $A$  the Stokes operator  $Au = -\Delta u$  for  $u \in D(A)$ . We recall that the operator  $A$  is a closed positive self-adjoint unbounded operator, with  $D(A) = \{u \in V_0, Au \in V_0\}$ . We have in fact,  $D(A) = V_2$ . Now define the trilinear form  $b(., ., .)$  associated with the inertia terms

$$b(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx \quad (2.1)$$

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2000 *Mathematics Subject Classification.* 35Q30, 35A02, 35D30.

*Key words and phrases.* Navier-Stokes equations - weak solutions - uniqueness.

The continuity property of the trilinear form enables us to define (using Riesz representation theorem) a bilinear continuous operator  $B(u, v); V \times V \rightarrow V'$  will be defined by

$$\langle B(u, v), w \rangle = b(u, v, w), \quad \forall w \in V. \quad (2.2)$$

Recall that for  $u$  satisfying  $\nabla \cdot u = 0$  we have

$$b(u, u, u) = 0 \text{ and } b(u, v, w) = -b(u, w, v). \quad (2.3)$$

For the 2D case and only with periodic boundary conditions, we have

$$b(u, u, Au) = 0. \quad (2.4)$$

Hereafter,  $c_i \in \mathbb{N}$ , will denote a dimensionless scale invariant positive constant which might depend on the shape of the domain. The trilinear form  $b(., ., .)$  is continuous on  $V_{m_1}(\Omega) \times V_{m_2+1}(\Omega) \times V_{m_3}(\Omega)$ ,  $m_i \geq 0$

$$|b(u, v, w)| \leq c_0 \|u\|_{m_1} \|v\|_{m_2+1} \|w\|_{m_3}, \quad m_3 + m_2 + m_1 \geq \frac{3}{2} \quad (2.5)$$

see [4]. We recall some inequalities that we will be using in what follows.

Young's inequality

$$ab \leq \frac{\sigma}{p} a^p + \frac{1}{q\sigma^{\frac{q}{p}}} b^q, \quad a, b, \sigma > 0, p > 1, q = \frac{p}{p-1}. \quad (2.6)$$

Poincaré's inequality

$$\lambda_1 \|u\|^2 \leq \|u\|_1^2 \text{ for all } u \in V_0, \quad (2.7)$$

where  $\lambda_1$  is the smallest eigenvalue of the Stokes operator  $A$ .

### 3. NAVIER-STOKES EQUATIONS

The conventional Navier-Stokes system can be written in the evolution form

$$\begin{aligned} \frac{\partial u}{\partial t} + \nu Au + B(u, u) &= f, \quad t > 0, \\ \operatorname{div} u &= 0, \quad \text{in } \Omega \times (0, \infty) \text{ and } u(x, 0) = u_0, \quad \text{in } \Omega, \end{aligned} \quad (3.1)$$

We recall that a Leray weak solution of the Navier-Stokes equations is a solution which is bounded and weakly continuous in the space of periodic divergence-free  $L^2$  functions, whose gradient is square-integrable in space and time and which satisfies the energy inequality. The proof of the following theorem is given in [2, 4, 8]

**Theorem 3.1.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n = 2, 3$  and  $f \in L^2(0, T; V_1')$ ,  $u_0 \in V_0$  be given. Then there exists a weak solution  $u$  of (3.1) which satisfies  $u \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0)$ ,  $\forall T > 0$ , Furthermore if  $n = 2$ ,  $u$  is unique.*

In this paper we will be especially interested in the case where  $n = 3$ . Consider the difference of two solutions,  $w = u - v$  and  $w_0 = 0$ . This satisfies the equation

$$\partial_t w + Aw + B(v, v) - B(u, u) = 0. \quad (3.2)$$

Taking the inner product with  $w$  we obtain

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 + \nu \|A^{\frac{1}{2}} w\|^2 = b(w, w, u). \quad (3.3)$$

Now, if we consider (2.2) with  $(m_3, m_2, m_1) = (0, 1/2, 1)$ , it follows that

$$|b(w, w, u)| \leq \frac{\nu}{2} \|A^{\frac{1}{2}} w\|^2 + \frac{\nu}{2} \|u\|^2 \|A^{\frac{3}{4}} w\|^2. \quad (3.4)$$

Combining all these inequalities in (3.3), we obtain that

$$\frac{d}{dt} \|w\|^2 + \frac{\nu}{2} \|w\|_1^2 \leq \frac{\nu}{2} \|u\|^2 \|A^{\frac{3}{4}} w\|^2. \quad (3.5)$$

Integrating the differential inequality, for  $t \geq 0$  we get

$$\int_0^t \|w\|_1^2 ds \leq \int_0^t \|u\|^2 \|A^{\frac{3}{4}} w\|^2 ds. \quad (3.6)$$

We take the inner product of (3.1) with  $u$ , we obtain

$$\frac{d}{dt} \|u\|^2 + 2\nu \|\nabla u\|^2 = 2(f, u). \quad (3.7)$$

Here we have used the fact that  $b(u, u, u) = 0$ . By applying Young's inequality and the Poincaré Lemma, we get

$$\frac{d}{dt} \|u\|^2 + \nu \|u\|^2 \leq \frac{1}{\nu} \|f\|_{V_1'}^2, \quad (3.8)$$

by integrating the above inequality from 0 to  $t$ , we get

$$\|u\|^2 \leq \|u_0\|^2 + \frac{1}{\nu} \int_0^t \|f\|_{V_1'}^2 ds, \quad (3.9)$$

then

$$\|u\|_{L^\infty(0,T;V_0)}^2 \leq \|u_0\|^2 + \frac{1}{\nu} \int_0^t \|f\|_{V_1'}^2 ds = R^2. \quad (3.10)$$

This means that

$$\int_0^t \|w\|_1^2 ds \leq R^2 \int_0^t \|A^{\frac{3}{4}} w\|^2 ds. \quad (3.11)$$

We will consider the problem of estimating a priori the-typical area of a level set of the function  $w$ . For any positive number  $\gamma$  we consider the level sets for the difference of an arbitrary two solutions of the 3D Navier Stokes equations

$$\mathfrak{S}_\gamma = \{x; w(x, t) = \gamma, t > 0\}, \quad (3.12)$$

We denote their 2-dimensional Hausdorff measure (area in this case) by  $\mu(\gamma, t)$

$$\mu(\gamma, t) = \int_{\mathfrak{S}_\gamma} dH^{(2)}(x) dx. \quad (3.13)$$

We are interested in estimating the average  $\langle \mu \rangle$ . We take an arbitrary a positive smooth function  $\varphi$  of one variable compactly supported in the interval  $(0, \infty)$  and satisfies the inequality

$$\varphi(x) \leq 1. \quad (3.14)$$

We observe that

$$\int_0^t \int \varphi^2(w) \|w\|_1^2 dx ds \leq R^2 \int_0^t \int \|A^{\frac{3}{4}} w\|^2 dx ds, \quad (3.15)$$

and, as a consequence

$$\int_0^t \int \varphi(w(x, t)) \|w\|_1 dx ds \leq R^2 \int_0^t \int \|A^{\frac{3}{4}} w\|^2 dx ds. \quad (3.16)$$

Now we will use an important tool of geometric measure theory, established by Kronrod [5] in a specific case and by Federer [3] in the general case. A proof can be found in [7] also.

**Theorem 3.2.** *Let  $\psi$  be a Borel measurable nonnegative function on  $D$  and let  $h$  be a Lipschitz continuous function in  $D$ , where  $D$  is an open subset of  $\mathbb{R}^n$ . Then*

$$\int_D \psi(x) |\nabla h(x)| dx = \int_0^\infty d\beta \int_{\mathcal{A}_\beta} \psi(x) dH^{n-1} dx, \quad (3.17)$$

where  $H^{n-1}$  is the  $(n-1)$ -dimensional Hausdorff measure and

$$\mathcal{A}_\beta = \{x \in D; h(x) = \beta\} \quad (3.18)$$

Theorem 3.2 is local and therefore it is valid for periodic functions, also. We will use it in (3.16), with  $h = w$ ,  $\psi(x) = \varphi(x)$  and  $\mathcal{A}_\beta = \mathfrak{S}_\gamma$ . We obtain

$$T^{-1} \int_0^T ds \int_0^\infty \varphi(\gamma) d\gamma \int_{\mathfrak{S}_\gamma} dH^{(2)}(x) dx \leq T^{-1} R^2 \int_0^T \|A^{\frac{3}{4}} w\|^2 ds \quad (3.19)$$

and therefore the inequality (3.19) can be re-written as

$$T^{-1} \int_0^T \int \varphi(\gamma) \mu(\gamma, t) d\gamma ds \leq T^{-1} R^2 \int_0^T \|A^{\frac{3}{4}} w\|^2 ds. \quad (3.20)$$

Let us define  $\mu_T$  by

$$\mu_T = T^{-1} \int_0^T \mu(\gamma, s) ds. \quad (3.21)$$

Because  $\varphi$  is arbitrary we obtain from (3.20) that

$$\int_0^\infty \mu_T^2(\gamma, s) d\gamma \leq T^{-1} R^2 \int_0^T \|A^{\frac{3}{4}} w\|^2 ds. \quad (3.22)$$

In view of the energy inequality (3.9) the inequality (3.22) implies the a priori bounds in terms of initial data  $u_0$  and  $f$

$$\int_0^\infty \mu_T^2(\gamma) d\gamma \leq (\|u_0\|^2 + \frac{1}{\nu} \int_0^t \|f\|_{V_1'}^2 ds) T^{-1} \int_0^T \|A^{\frac{3}{4}} w\|^2 ds. \quad (3.23)$$

In a completely analogous fashion, passing to  $\limsup_{T \rightarrow \infty}$  in (3.23) we obtain

$$\int_0^\infty \mu_\infty^2(\gamma) d\gamma \leq \limsup_{T \rightarrow \infty} (T^{-1} R^2 \int_0^T \|A^{\frac{3}{4}} w\|^2 ds) \quad (3.24)$$

where

$$\mu_\infty(\gamma) = \limsup_{T \rightarrow \infty} T^{-1} \int_0^T \mu_T(\gamma) d\gamma \quad (3.25)$$

and

$$\int_0^\infty \mu_\infty^2(\gamma) d\gamma \leq (\|u_0\|^2 + \frac{1}{\nu} \|f\|_{L^2(0, \infty; V_1')}^2) \limsup_{T \rightarrow \infty} T^{-1} \int_0^T \|A^{\frac{3}{4}} w\|^2 ds. \quad (3.26)$$

We recall that the Kolmogorov's mean rate of dissipation of energy  $\eta$  in  $2D$  turbulent flow (see e.g. [4]) is defined as

$$\eta = \frac{1}{2} \lambda_1 \nu \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|Au\|^2 ds. \quad (3.27)$$

In order to measure the difference between the velocity fields throughout the set  $\mathfrak{S}_\gamma$ , we introduce the following quantity, which is defined for each velocity field  $w$  by

$$\eta_\bullet = \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|A^{\frac{3}{4}} w\|^2 ds, \quad (3.28)$$

for instance the quantity  $\eta_{\bullet}$  denote the enstrophy dissipation rate of  $w = u - v$ . The inequality (3.26) becomes

$$\int_0^\infty \mu_\infty^2(\gamma) d\gamma \leq cR_\infty^2 \eta_{\bullet}, \quad (3.29)$$

where

$$R_\infty^2 = \|u_0\|^2 + \frac{1}{\nu} \|f\|_{L^2(0,\infty;V_1')}^2$$

Note that the right-hand sides of (3.23) and (3.26) are finite even if the energy becomes singular. We proved thus

**Theorem 3.3.** *Let  $w$  be the the difference of two classical  $L$ -periodic solution of the Navier-Stokes equation (2.1) with smooth periodic driving forces  $f$ . Assume that the solutions are defined for  $0 < t < T_1$ , where  $T_1 < C_o$ . Then the area*

$$\mu(\gamma, t) = \int_{\{x; w(x,t)=\gamma\}} dH^{(2)}(x) dx \quad (3.30)$$

satisfies, for any  $T \leq T_1$

$$\int_0^\infty \mu_T^2(\gamma) d\gamma \leq R^2 T^{-1} \int_0^T \|A^{\frac{3}{4}} w\|^2 dt \quad (3.31)$$

where  $\mu_T$  is the time average of  $\mu$  defined in (3.21). Moreover, if  $T = \infty$  then

$$\int_0^\infty \mu_\infty^2(\gamma) d\gamma \leq cR_\infty^2 \eta_{\bullet}, \quad (3.32)$$

where  $\mu_\infty$  is  $\limsup_{T \rightarrow \infty} \mu_T$ .

Since  $R$  and  $R_\infty$  are independent of the time, the uniqueness depends only on the new quantities  $\eta_{\bullet}$ .

**Proposition 3.4.** *Let  $\delta_\infty$  be as in (3.42) below. Then the 2-dimensional Hausdorff measure  $\mu(\gamma, t)$  satisfies:*

*if  $\delta_\infty < \nu$ , then*

$$\mu \leq 0. \quad (3.33)$$

*If  $\delta_\infty > \nu$ , then*

$$\mu = 0. \quad (3.34)$$

*If  $\delta_\infty \neq \nu$  and  $\|A^{\frac{1}{4}} w_0\| = 0$ , then*

$$\mu = 0. \quad (3.35)$$

*For  $\delta_\infty = \nu$ ,  $\mu$  is infinite.*

*Proof.* If we take the inner product of both sides of (3.2) with  $A^{\frac{1}{2}} w$  we obtain

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{4}} w\|^2 + \nu \|A^{\frac{3}{4}} w\|^2 + B(w, v, A^{\frac{1}{2}} w) + B(u, w, A^{\frac{1}{2}} w) = 0, \quad (3.36)$$

or

$$B(u, w, A^{\frac{1}{2}} w) \leq \|u\| \|A^{\frac{3}{4}} w\|^2 \quad (3.37)$$

and

$$B(w, v, A^{\frac{1}{2}} w) \leq \|v\| \|A^{\frac{3}{4}} w\|^2, \quad (3.38)$$

then we have

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{4}} w\|^2 + \nu \|A^{\frac{3}{4}} w\|^2 \leq (\|u\| + \|v\|) \|A^{\frac{3}{4}} w\|^2. \quad (3.39)$$

We denote

$$(\|u\| + \|v\|) = \delta, \quad (3.40)$$

applying this to (3.39) we have that

$$\|A^{\frac{1}{4}}w\|^2 + \nu \int_0^t \|A^{\frac{3}{4}}w\|^2 \leq \delta_\infty \int_0^t \|A^{\frac{3}{4}}w\|^2 + \|A^{\frac{1}{4}}w_0\|^2 \quad (3.41)$$

where

$$\delta_\infty = \sup_{t \in [0, \infty]} \delta \leq 2R_\infty \quad (3.42)$$

then from (3.41) we obtain that

$$\int_0^t \|A^{\frac{3}{4}}w\|^2 \leq \frac{1}{(\nu - \delta_\infty)} \|A^{\frac{1}{4}}w_0\|^2. \quad (3.43)$$

This concludes the proof of Theorem 3.4.  $\square$

In order to apply these results to the  $2D$  periodic Navier Stokes equations, where their uniqueness are well known as classical results of Leray [6]. We study the one-dimensional Hausdorff measure

$$\mu(\gamma, t) = \int_{\{x; w(x, t) = \gamma\}} dH^{(1)}(x) dx \quad (3.44)$$

operating on subsets of  $\mathbb{R}^2$ . We also call it the Hausdorff length of the level sets of the difference between two solutions.

**Proposition 3.5.** *Let  $w$  be the the difference of two classical  $L$ -periodic solution of the  $2D$  Navier-Stokes equation. Assume that the solutions are defined for  $0 < t < T$ . Then the length of  $\mathfrak{S}_\gamma$  satisfies,*

$$\int_0^\infty \mu_T(\gamma) d\gamma \leq R_1^2 T^{-1} \int_0^T \|w\|^2 dt \quad (3.45)$$

where  $\mu_T$  is the time average of  $\mu$  defined in (3.44) and

$$R_1^2 = \|u\|_{L^\infty(0, T; V_1)}^2 \leq \|\nabla u_0\|^2 + \frac{1}{\nu} \int_0^T \|f\|^2 ds. \quad (3.46)$$

*Proof.* In two dimensions, inequality (3.3) becomes

$$\frac{d}{dt} \|w\|^2 + \nu \|w\|_1^2 \leq \frac{c_1}{\nu} \|\nabla u\|^2 \|w\|^2. \quad (3.47)$$

We take the inner product of (3.1) with  $Au$ , we obtain

$$\frac{d}{dt} \|\nabla u\|^2 + 2\nu \|Au\|^2 = 2(f, Au). \quad (3.48)$$

Here we have used the fact that  $b(u, u, Au) = 0$ . By applying Young's inequality, we get

$$\frac{d}{dt} \|\nabla u\|^2 + \nu \|Au\|^2 \leq \frac{\|f\|^2}{\nu}, \quad (3.49)$$

by integrating the above inequality from 0 to  $T$ , we get

$$\|u\|_1^2 \leq \|u_0\|_1^2 + \frac{1}{\nu} \int_0^T \|f\|^2 ds. \quad (3.50)$$

This gives

$$\|u\|_{L^\infty(0,T;V_1)}^2 \leq \frac{1}{\nu} \int_0^T \|f\|^2 ds + \|u_0\|_1^2 = R_1^2. \quad (3.51)$$

It follows from (3.47) that

$$\int_0^t \|w\|_1^2 ds \leq R_1^2 \int_0^t \|w\|^2 ds. \quad (3.52)$$

The rest of the proof of the proposition is analog to that of Theorem 3.3.  $\square$

It is clear that  $\mu(0, t) = 0$ . This particular result is so important and it is so in accordance with the classical uniqueness result. By using the Poincaré inequality, this upper bound can be expressed in terms of the energy dissipation rate  $\varepsilon_\blacktriangle$  or in terms of the enstrophy dissipation rate  $\eta_\blacktriangle$ . As a continuation of the previous work [9, 10], we introduce in this paper new directions for the study of the uniqueness and give a new link between the uniqueness, geometric and energy in the Navier Stokes equations.

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